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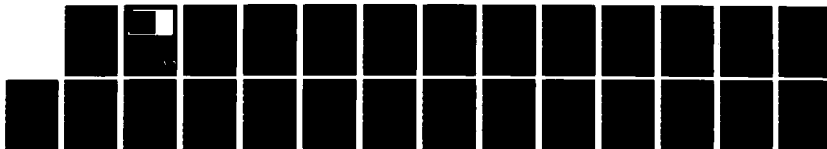
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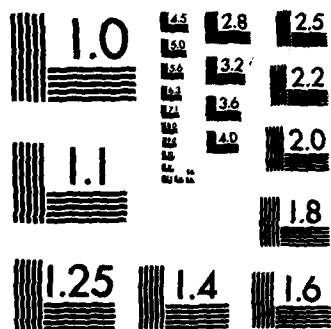
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ON A VOLTERRA EQUATION WITH A LOCALLY
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ON A VOLTERRA EQUATION WITH A LOCALLY FINITE MEASURE AND L^∞ -PERTURBATION

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ABSTRACT

Let $g(x)$ be locally Lipschitzian, $f \in L^1_{loc}(R^+)$ and let μ be a real locally finite positive definite Borel measure on R^+ . We investigate the asymptotics of the solutions of the nonlinear scalar Volterra equation

$$x'(t) + \int_{[0,t]} g(x(t-s))d\mu(s) = f(t), \quad t \in R^+, \quad x(0) = x_0$$

in the case when f only satisfies $\lim_{t \rightarrow \infty} (r * f)(t) = 0$. The function $r(t)$ is defined as the solution of

$$r'(t) + \int_{[0,t]} r(t-s)d\mu(s) = 0, \quad t \in R^+, \quad r(0) = 1,$$

and we assume $r' \in (L^1 \cap NBV)(R^+)$. No sign condition is imposed on $g(x)$.

AMS(MOS) Subject Classifications: 45D05, 45M05, 45G10

Key Words: Volterra equations, Nonlinear integral equations, Asymptotic behavior, Frequency domain methods

Work Unit Number 1 (Applied Analysis)

SIGNIFICANCE AND EXPLANATION

In the construction of mathematical models of technical and physical systems one is frequently led to equations in which the current rate of change ($= \frac{dx}{dt}$) of the state of the system ($= x(t)$ at time t) is a function not only of $x(t)$, but also of $x(\tau)$ for past times $\tau < t$. Specifically, one obtains Volterra integrodifferential equations, exemplified by

$$(E) \quad \frac{dx}{dt} + \int_0^t g(x(t-s))d\mu(s) = f(t), \quad x(0) = x_0, \quad t > 0.$$

Here $f(t)$ is the external input, $\mu(t)$ is the feedback kernel, $g(x)$ is in general a nonlinear function of x . By letting $\mu(t)$ have discontinuities we realize that (E) includes a large class of differential-delay equations.

The key problem concerning (E) is the behavior of $x(t)$ for large values of t . In particular one is interested in whether the solutions $x(t)$ remain bounded and in case they do, whether $x(t)$ tends to a limit when $t \rightarrow \infty$, or whether the system continues to oscillate. The present report analyzes these questions and continues work begun in MRC Technical Summary Report #2224. We are in particular interested in the case when the variation of the feedback kernel is large, in the sense that $\mu(t)$ is not of bounded variation over the positive half-axis. Such kernels are frequent in applications; let for example $d\mu(s) = b(s)ds$ with $b(s) = (\cos s)s^{-\alpha}$, $0 < \alpha < 1$. The second key feature of this report is that we do allow large input functions $f(t)$ in that we only assume $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

One main result (Theorem 2) follows from four auxiliary results, all of which are established in this report. The first shows that a solution of (E) cannot oscillate arbitrarily much; in particular there do not exist positive constants δ, T such that

$$\int_{t-T}^t |x'(\tau)| d\tau > \delta$$

holds for all sufficiently large t . The second is a rather technical result comparing the size of $\int_{I_n} g^2(x(\tau))d\tau$ to that of $\int_{I_n} x^2(\tau)d\tau$, where I_n is a sequence of longer and longer intervals. The thirdⁿ shows that solutions y of a particular limit equation associated with (E) satisfy

$\int_{-\infty}^{\infty} |g(y(\tau))|^2 d\tau < \infty$. Finally, in the last auxiliary result we show that a certain energy function decreases along solutions of (E).

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON A VOLTERRA EQUATION WITH A LOCALLY FINITE MEASURE AND L^∞ -PERTURBATION

Stig-Olof Londen

1. INTRODUCTION

In this report we continue the analysis of the asymptotics of the scalar, real, nonlinear Volterra equation with a positive definite measure μ ,

$$(1.1) \quad x'(t) + \int_{[0,t]} g(x(t-s)) d\mu(s) = f(t), \quad t \in \mathbb{R}^+, \quad x(0) = x_0,$$

in the case when both μ and f are large in a sense to be made precise below. This analysis was begun in [2] where the following key result was obtained.

Theorem 1 [2, Theorem 4]. Let

(1.2) μ be a locally finite, positive definite measure on \mathbb{R}^+ , satisfying

$$(1.3) \quad \text{Im } \hat{\mu}(\omega) = 0 \text{ for } \omega \in \{\omega | \omega \neq 0, \text{Re } \hat{\mu}(\omega) = 0\}.$$

Define $r(t)$, $t \in \mathbb{R}^+$, as the locally absolutely continuous solution of

$$(1.4) \quad r'(t) + (r * \mu)(t) = 0, \text{ a.e. on } \mathbb{R}^+, \quad r(0) = 1,$$

and suppose

$$(1.5) \quad r' \in (L^1 \cap NBV)(\mathbb{R}^+),$$

$$(1.6) \quad f \in L^1_{loc}(\mathbb{R}^+),$$

$$(1.7) \quad \lim_{t \rightarrow \infty} r(t) = \lim_{t \rightarrow \infty} (r * f)(t) = 0.$$

Also let

(1.8) $g(x)$ be locally Lipschitzian, $x \in \mathbb{R}$,

$$(1.9) \quad \liminf_{|x| \rightarrow 0} x^{-1} g(x) > 0,$$

$$(1.10) \quad xg(x) > 0, \quad x \neq 0.$$

Finally suppose that

$$(1.11) \quad x \in (LAC \cap L^\infty)(\mathbb{R}^+)$$

satisfies (1.1) a.e. on \mathbb{R}^+ . Then

$$(1.12) \quad \lim_{t \rightarrow \infty} x(t) = 0.$$

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Above $\hat{\mu}(\omega)$, $\omega \neq 0$, is defined as $\lim_{\substack{s+i\omega \\ \text{Re } s > 0}} \int_{\mathbb{R}^+} e^{-st} d\mu(t)$. By (1.5) this is well-

defined, although possibly infinite (see [2] for details). By $(r * \mu)(t)$ we mean

$\int_{[0,t]} r(t-s) d\mu(s)$ and the convolution $(f_1 * f_2)(t)$ of two functions $f_1, f_2 \in L^1_{loc}(\mathbb{R}^+)$ is defined as $\int_0^t f_1(t-s) f_2(s) ds$.

The above result shows that if the solution $z(t) = x_0 r(t) + (r * f)(t)$ of the linear equation with the same data as (1.1), namely

$$z'(t) + (z * \mu)(t) = f(t), \quad t \in \mathbb{R}^+, \quad z(0) = x_0,$$

vanishes at infinity then so does, under very weak conditions on μ and f , a bounded solution of the nonlinear equation (1.1). Certain assumptions, namely (1.8)-(1.10), are however imposed on the nonlinearity g . The first, which requires g to be locally Lipschitzian, is essential to the proof. The second assumption is of a more technical nature. None of these two assumptions is overly restrictive.

The sign condition (1.10) does however impose a certain restriction on the class of nonlinearities to which Theorem 1 applies and obviously the removal of this assumption would strengthen the result appreciably. As moreover no assumption of this type is needed if the size of μ or of f is radically decreased; that is if $\int_{\mathbb{R}^+} d|\mu|(t) < \infty$ or $f \in L^1(\mathbb{R}^+)$ is assumed, then one is strongly motivated to try to remove (1.10). That this can in fact be done is shown by the following

Theorem 2. Let (1.2), (1.4)-(1.8) and (1.11) hold. In addition suppose that

$$(1.13) \quad \text{Re } \hat{\mu}(\omega) > 0, \quad \omega \neq 0, \quad \inf_{\omega \in S, \omega \neq 0} \text{Re } \hat{\mu}(\omega) > 0,$$

for any compact set $S \subset \mathbb{R}$. Define $X = \{y \in \mathbb{R} \mid |y| < |x|, \quad x \in L^1(\mathbb{R}^+), \quad g(y) = 0\}$ and assume that

X consists of a finite number of points, $X = \{\alpha_1, \dots, \alpha_n\}$. Finally let

$$(1.14) \quad \liminf_{|x-\alpha_1| \rightarrow 0} |g(x)| |x - \alpha_1|^{-1} > 0, \quad \forall i.$$

Then

$$(1.15) \quad \lim_{t \rightarrow \infty} \text{dist}(x(t), X) = 0.$$

Further generalization as to the size of X is possible. Also note that (1.14) of course corresponds to (1.9). The removal of (1.10) has however forced us to strengthen (1.3) to (1.13). But this is a minor strengthening compared to the absence of (1.10).

The proof of Theorem 2 is divided into four lemmas and some final arguments. Lemmas 1 and 2 build upon and continue estimates of [2, Theorem 2]. We note that Lemma 1 follows from Lemma 2 if (1.14) is assumed. But (1.14) is neither used to prove Lemma 1 nor for the proof of Lemma 2. Only for the proof of Lemma 3 is this condition needed. Therefore we prefer to state Lemmas 1 and 2 separately.

Lemma 3 is a straightforward consequence of Lemma 2 and of (1.14). The final Lemma 4 shows that the energy function $G(x)$ strictly decreases along those nontrivial solutions y of a certain limit equation which satisfy $y(=\infty), y(-\infty) \in X$. To prove Lemma 4 a fairly elaborate use is made of the differential resolvents r_λ , $\lambda > 0$, which satisfy

$$r'_\lambda(t) + \lambda(r_\lambda * \mu)(t) = 0, \quad r_\lambda(0) = 1,$$

after which one lets $\lambda \rightarrow 0$.

For additional comments and related results we refer the reader to [2].

2. PROOF OF THEOREM 2

Convolve (1.1) by $r(t)$, use (1.4) and define

$$h(x) = g(x) - x, \quad x \in R; \quad a(t) = -r'(t), \quad t \in R^+.$$

This gives

$$x(t) + \int_0^t h(x(t-s))a(s)ds = x_0 r(t) + (r * f)(t), \quad t \in R^+.$$

Let $L^+(x)$ be the positive limit set of $x(t)$, thus

$$L^+(x) = \{y | y \in (L^\infty \cap LAC)(R), \exists t_n \rightarrow \infty \text{ such that } x(t_n + t) \rightarrow y(t), \\ \text{as } n \rightarrow \infty, \text{ uniformly on compact sets of } R\}.$$

As $h \in C(R)$, $a \in L^1(R^+)$ and as $x_0 r(\infty) + (r * f)(\infty) = 0$ it follows that any $y \in L^+(x)$ satisfies

$$(2.1) \quad y(t) + \int_{R^+} h(y(t-s))a(s)ds = 0, \quad t \in R,$$

and also

$$(2.2) \quad y'(t) + \int_{R^+} h(y(t-s))d\alpha(s) = 0, \quad \text{a.e. on } R,$$

where the finite measure α is defined by $\alpha([0, t]) = a(t)$.

Let $\hat{\alpha}, \hat{a}$ be defined by

$$\hat{\alpha}(\omega) = \int_{R^+} e^{-i\omega t} d\alpha(t), \quad \hat{a}(\omega) = \int_{R^+} e^{-i\omega t} a(t)dt.$$

Straightforward computations using (1.4) give

$$(2.3) \quad \operatorname{Re} \hat{\alpha}(\omega) = \omega^2 \operatorname{Re} \hat{\mu}(\omega) |i\omega + \hat{\mu}(\omega)|^{-2}, \quad \omega \neq 0,$$

$$(2.4) \quad \operatorname{Im} \hat{\alpha}(\omega) = \omega^2 \operatorname{Im} \hat{\mu}(\omega) |i\omega + \hat{\mu}(\omega)|^{-2} + \omega |\hat{\mu}(\omega)|^2 |i\omega + \hat{\mu}(\omega)|^{-2}, \quad \omega \neq 0,$$

$$(2.5) \quad \hat{\alpha}(0) = 0,$$

$$(2.6) \quad \hat{a}(\omega) = \hat{\mu}(\omega) [i\omega + \hat{\mu}(\omega)]^{-1}, \quad \omega \neq 0; \quad \hat{a}(0) = 1.$$

As μ is positive definite we have $\operatorname{Re} \hat{\mu} > 0$, $\omega \in R \setminus \{0\}$, and so $\operatorname{Re} \hat{\alpha}(\omega) > 0$, $\omega \in R$.

Also note that by (1.13), (2.3), (2.5)

$$\operatorname{Re} \hat{\alpha}(\omega) = 0, \quad \omega \in Z \stackrel{\text{def}}{=} Z_1 \cup \{0\},$$

where $Z_1 \stackrel{\text{def}}{=} \{\omega | \omega \neq 0, |\hat{u}(\omega)| = \infty\}$. Note that $\text{Im } \hat{a}(\omega) = \omega \neq 0$ for $\omega \in Z_1$ and that $\hat{a}(\omega) = 1$ if and only if $\omega \in Z$. Thus, as $a \in L^1(\mathbb{R}^+)$, it follows that Z is compact.

Our first goal is to prove

LEMMA 1. There does not exist $y \in L^+(x)$ such that for some positive constants δ, T

$$(2.7) \quad \int_{t-T}^t |y'(s)| ds > \delta, \quad t \in \mathbb{R}.$$

Proof of Lemma 1. For $t > 0$, let

$$m_t^2 \stackrel{\text{def}}{=} \sup_{y \in L^+(x)} \|h(y(\tau))\|_{L^2(0,t)}^2.$$

Assume that there exists $y \in L^+(x)$ such that (2.7) holds for some positive δ, T . Take any such y . Then by (2.2) and as u is finite,

$$(2.8) \quad m_t^2 \leq \rho \|h(y(\tau))\|_{L^2(0,t)}^2$$

for some positive constant ρ .

Fix $\hat{\epsilon} > 0$ arbitrarily, let $\omega_0 > 0$ be such that

$$(2.9) \quad |1 - \hat{a}(\omega)| < \hat{\epsilon}, \quad |\omega| < \omega_0,$$

then take ϵ sufficiently small and define

$$(2.10) \quad S_1 = \{\omega | \text{dist}(\omega, Z_1) < \epsilon, |1 - \hat{a}(\omega)| < \hat{\epsilon}, \omega_0 < |\omega|\},$$

$$(2.11) \quad S = S_1 \cup [-\omega_0, \omega_0].$$

Introduce truncated functions u_t, v_t, z_t as follows:

$$(2.12) \quad \begin{cases} u_t(\tau) = y'(\tau), & v_t(\tau) = y(\tau), & z_t(\tau) = h(y(\tau)), & \tau \in [0, t] \\ u_t(\tau) = v_t(\tau) = z_t(\tau) = 0, & \tau < 0, \tau > t. \end{cases}$$

A comparison of the present Theorem with [2, Theorem 2] reveals that all the assumptions made in [2, Theorem 2] are now satisfied except $xg(x) > 0, x \in \mathbb{R}$, which is now absent from the assumptions. As y satisfies (2.8) - compare with [2, relation (3.2)] - we may therefore use all those estimates of the proof of [2, Theorem 2] which were obtained without making use of the sign condition on g . In particular we have, (see (3.30) of [2]), if T is fixed sufficiently large,

$$(2.13) \quad \int_{\mathbb{R} \setminus S} |\hat{u}_t|^2 d\omega < \epsilon \hat{c}_1 \int_S |\hat{z}_t|^2 d\omega + K_1(\epsilon, T), \quad t > T,$$

where the constant \hat{c}_1 depends on $\hat{\epsilon}$, but not on ϵ, t, T and where K_1 is a constant depending on ϵ, T .

Let

$$(2.14) \quad F(\tau) \stackrel{\text{def}}{=} y(\tau) - \int_0^\tau y(\tau-s)a(s)ds, \quad \tau > 0; \quad F(\tau) \stackrel{\text{def}}{=} 0, \quad \tau < 0.$$

We wish to show that there exist intervals $I_n \subset \mathbb{R}^+$, $m(I_n) > 0$, such that

$\limsup_{n \rightarrow \infty} \sup_{\tau \in I_n} |F(\tau)| = 0$. To this end define F_t by

$$(2.15) \quad F_t(\tau) = v_t(\tau) - \int_{\mathbb{R}^+} v_t(\tau-s)a(s)ds, \quad \tau \in \mathbb{R}.$$

Then $F(\tau) = F_t(\tau)$, $\tau < t$ and $F_t(\tau) = - \int_{\tau-t}^\tau y(\tau-s)a(s)ds$, $\tau > t$. For fixed t $F_t \in (L^1 \cap L^\infty)(\mathbb{R})$ which implies that the Fourier transforms to follow are well defined. A transformation of (2.15) gives $\hat{F}_t = \hat{v}_t[1 - \hat{a}]$ and so, from (2.9)-(2.11),

$$(2.16) \quad \int_S |\hat{F}_t(\omega)|^2 d\omega < \epsilon^2 \int_S |\hat{v}_t(\omega)|^2 d\omega < \epsilon^2 \int_R |\hat{v}_t(\omega)|^2 d\omega,$$

$$(2.17) \quad \int_{R \setminus S} |\hat{F}_t(\omega)|^2 d\omega < c_0^2 \int_{R \setminus S} |\hat{v}_t(\omega)|^2 d\omega \quad c_0 \stackrel{\text{def}}{=} \sup_{\omega \in R} |1 - \hat{a}(\omega)|.$$

From (2.12) it follows that $(c \stackrel{\text{def}}{=} \|x\|_{L^1(\mathbb{R}^+)})$

$$(2.18) \quad |\hat{v}_t(\omega)| < 2c|\omega|^{-1} + |\omega^{-1}\hat{u}_t(\omega)|.$$

Consequently, after squaring both sides of (2.18), integrating over $R \setminus S$ and recalling

(2.13) one gets

$$(2.19) \quad \int_{R \setminus S} |\hat{v}_t(\omega)|^2 d\omega < 4c^2\omega_0^{-1} + 2\omega_0^{-2} \int_{R \setminus S} |\hat{u}_t(\omega)|^2 d\omega < 2\epsilon c_1\omega_0^{-2} \int_S |\hat{z}_t(\omega)|^2 d\omega + K_2$$

where $K_2 \stackrel{\text{def}}{=} 4c^2\omega_0^{-1} + 2K_1\omega_0^{-2}$. But by Parseval's relation and by (1.8) we have

$$(2.20) \quad \int_S |\hat{z}_t|^2 d\omega < \int_R |\hat{z}_t|^2 d\omega < \lambda^2 \int_R |\hat{v}_t|^2 d\omega$$

where λ is the local ($|y| < c$) Lipschitz constant of $h(y)$. The relations (2.19),

(2.20) imply

$$(2.21) \quad \int_{R \setminus S} |\hat{v}_t|^2 d\omega < c c_2 \int_R |\hat{v}_t|^2 d\omega + K_2$$

where $\hat{c}_2 = 2\omega_0^{-2} \lambda^2 c_1$. By (2.16), (2.17) and by (2.21)

$$\int_R |\hat{F}_t|^2 d\omega < [\hat{c}^2 + c c_2 c_0^2] \int_R |\hat{v}_t|^2 d\omega + c_0^2 K_2.$$

But $\int_R |\hat{v}_t|^2 d\omega < 2\omega c^2 t$ and so, as \hat{c}, c were arbitrary

$$\int_0^t |F(\tau)|^2 d\tau = \int_0^t |F_t(\tau)|^2 d\tau < \int_R |F_t(\tau)|^2 d\tau = \sigma(t), \quad t \rightarrow \infty.$$

Therefore, as also $F' \in L^\infty(R)$, we have that there exist intervals $I_n \subset R^+$, $m(I_n) \rightarrow \infty$ such that

$$(2.22) \quad \limsup_{n \rightarrow \infty} \sup_{\tau \in I_n} |F(\tau)| = 0.$$

From (2.14), (2.22), the uniform continuity of y and from the fact that $a \in L^1(R^+)$ it follows that there exists a subsequence $\{n_K\} \subset \{n\}$ and intervals $\tilde{I}_{n_K} = [\tilde{t}_{n_K}, t_{n_K}] \subset I_{n_K}$ such that $m(\tilde{I}_{n_K}) \rightarrow \infty$ as $n_K \rightarrow \infty$, and such that for some w satisfying

$$(2.23) \quad w(\tau) - \int_{R^+} w(\tau - s) a(s) ds = 0, \quad \tau \in R,$$

we have

$$y(\tau + \frac{1}{2}(\tilde{t}_{n_K} + t_{n_K})) \rightarrow w(\tau), \quad n_K \rightarrow \infty,$$

where the convergence is uniform on compact sets. But y (and hence any translate of y) satisfies (2.1) and consequently

$$(2.24) \quad w(\tau) + \int_{R^+} h(w(\tau - s)) a(s) ds = 0, \quad \tau \in R.$$

From (2.23), (2.24) one obtains

$$\int_{R^+} [h(w(\tau - s)) + w(\tau - s)] a(s) ds = 0, \quad \tau \in R,$$

and so [1, p. 297],

$$\sigma(h(w) + w) \subset \{\omega | \hat{a}(\omega) = 0\}.$$

But by (1.13) and (2.6) $\hat{a}(\omega) \neq 0$, $\omega \in R$, and hence $w(t) + h(w(t)) \equiv 0$, $t \in R$. Thus $g(w(t)) \equiv 0$, $t \in R$, which implies $w(t) \equiv \alpha_i$ for some i and

$$(2.25) \quad y(\tau + \frac{1}{2}(\tilde{t}_{n_K} + t_{n_K})) \rightarrow \alpha_i, \text{ as } n_K \rightarrow \infty,$$

uniformly for τ in compact sets. A combination of (2.2), (2.25) and the fact that $a(R^+) = 0$ does however violate (2.7). This contradiction completes the proof of Lemma 1.

In the next lemma we show that the size of $g(y(t))$ is small compared to that of $y(t)$.

LEMMA 2. Let $\{y_n\}_{n=1}^\infty \subset L^+(x)$ satisfy $h(y_n(t)) \notin L^2(R)$, let $t_n \rightarrow \infty$ and suppose that for every n and some $\rho > 0$ independent of n we have

$$(2.26) \quad \sup_{-\infty < s < t_n} \int_{s-2t_n}^s |h(y_n(\tau))|^2 d\tau < \rho \int_{-t_n}^{t_n} |h(y_n(\tau))|^2 d\tau.$$

Then for any $\varepsilon_0 > 0$, if n is sufficiently large,

$$\int_{-t_n}^{t_n} |g(y_n(\tau))|^2 d\tau < \varepsilon_0^2 \int_{-t_n}^{t_n} |h(y_n(\tau))|^2 d\tau.$$

Proof of Lemma 2. Define v_n, z_n, β_n by

$$v_n(\tau) = y_n(\tau), \quad z_n(\tau) = h(y_n(\tau)), \quad |\tau| < t_n, \quad v_n(\tau) = z_n(\tau) = 0, \quad |\tau| > t_n,$$

$$\beta_n(\tau) = v_n(\tau) + z_n(\tau), \quad \tau \in R.$$

Then, from (2.1)

$$(2.27) \quad \int_R \beta_n(t-s)a(s)ds = \int_R v_n(t-s)a(s)ds - v_n(t) + F_n(t), \quad t \in R,$$

where

$$F_n(t) = \begin{cases} 0, & t < -t_n, \\ -\int_{t+t_n}^t h(y_n(t-s))a(s)ds, & |t| \leq t_n, \\ \int_{t-t_n}^{t+t_n} h(y_n(t-s))a(s)ds, & t > t_n, \end{cases}$$

and $F_n \in (L^1 \cap L^\infty)(\mathbb{R})$.

Multiply (2.27) by β_n , integrate over \mathbb{R} and apply Parseval's relation. This gives

$$(2.28) \quad \int_{\mathbb{R}} |\hat{\beta}_n(\omega)|^2 \operatorname{Re} \hat{a}(\omega) d\omega = \int_{\mathbb{R}} \hat{\beta}_n(\omega) \hat{v}_n(\omega) [\hat{a}(\omega) - 1] d\omega + \int_{\mathbb{R}} \hat{F}_n(\omega) \hat{\beta}_n(\omega) d\omega.$$

Our purpose is now to estimate the right side of (2.28). As in the proof of Lemma 1 at first fix $\varepsilon > 0$ arbitrarily, take ω_0 such that (2.9) holds, then choose any $\varepsilon > 0$ and let S_1, S be defined as in (2.10), (2.11). For the first term I_1 on the right side of (2.28) we have (c_0 as in (2.17))

$$(2.29) \quad \begin{aligned} |I_1| &\leq \left(\int_{\mathbb{R}} |\hat{\beta}_n|^2 d\omega \right)^{1/2} \left(\int_{\mathbb{R}} |\hat{v}_n|^2 |\hat{a} - 1|^2 d\omega \right)^{1/2} \\ &\leq c_0 \left(\int_{\mathbb{R}} |\hat{\beta}_n|^2 d\omega \right)^{1/2} \left[\left(\int_{\mathbb{R} \setminus S} |\hat{v}_n|^2 d\omega \right)^{1/2} + \left(\int_S |\hat{v}_n|^2 |\hat{a} - 1|^2 d\omega \right)^{1/2} \right]. \end{aligned}$$

As $v_n(t) + \int_{\mathbb{R}} z_n(t-\tau)a(\tau)d\tau = F_n(t)$, $t \in \mathbb{R}$, then

$$(2.30) \quad |\hat{v}_n|^2 \leq 2c_1 |\hat{z}_n|^2 + 2|\hat{F}_n|^2,$$

where $c_1 \stackrel{\text{def}}{=} \sup_{\omega \in \mathbb{R}} |\hat{a}(\omega)|^2$. Because of (2.26) the same comments which were made before the relation (2.13) concerning the use of the proof of [2, Theorem 2] are still valid.

Therefore, if T is fixed sufficiently large, see [2, relation (3.25)],

$$(2.31) \quad \int_{\mathbb{R}} |\hat{F}_n|^2 d\omega \leq f_1(T) \int_{\mathbb{R}} |\hat{z}_n|^2 d\omega + f_2(T), \quad t_n > T,$$

where $\lim_{T \rightarrow \infty} f_1(T) = 0$, $f_2 \in L_{\text{loc}}^\infty(\mathbb{R}^+)$. Integrate (2.30) over S and use (2.31). The result is

$$(2.32) \quad \int_S |\hat{v}_n|^2 d\omega \leq 2[c_1 + f_1(T)] \int_{\mathbb{R}} |\hat{z}_n|^2 d\omega + 2f_2(T).$$

by (2.9)-(2.11) and by (2.32)

$$(2.33) \quad \left(\int_S |\hat{v}_n|^2 |\hat{a} - 1|^2 d\omega \right)^{1/2} < \hat{\epsilon} \left(\int_S |\hat{v}_n|^2 d\omega \right)^{1/2} < \\ \hat{\epsilon} [2c_1 + 2f_1]^{1/2} \left(\int_R |\hat{z}_n|^2 d\omega \right)^{1/2} + (2f_2)^{1/2} .$$

Now use (2.19) - with t replaced by n - and (2.33) to estimate the right side of (2.29)

upwards. This gives, for sufficiently large n , (assuming $\lim_{n \rightarrow \infty} \int_R |\hat{z}_n|^2 d\omega = \infty$)

$$|I_1| < c_2 [\hat{c}_3 \epsilon^{1/2} + \hat{\epsilon}] \left(\int_R |\hat{\beta}_n|^2 d\omega \right)^{1/2} \left(\int_R |\hat{z}_n|^2 d\omega \right)^{1/2} ,$$

where the constant c_2 is independent of $\epsilon, \hat{\epsilon}$ and where \hat{c}_3 is independent of ϵ .

For the second term I_2 on the right side of (2.28) we have, by (2.31)

$$|I_2| < (f_1(T) \int_R |\hat{z}_n|^2 d\omega + f_2(T))^{1/2} \left(\int_R |\hat{\beta}_n|^2 d\omega \right)^{1/2} .$$

Thus, if T is fixed sufficiently large,

$$(2.34) \quad |I_1| + |I_2| < 2c_2 [\hat{c}_3 \epsilon^{1/2} + \hat{\epsilon}] \|\hat{\beta}_n\|_{L^2(R)} \|\hat{z}_n\|_{L^2(R)}$$

for n such that $t_n > T$.

Having the estimate (2.34) for the right side of (2.28) our next goal is to obtain

(2.39). Recall that $\beta_n = v_n + z_n$, hence $|\hat{\beta}_n|^2 - |\hat{z}_n|^2 = |\hat{v}_n|^2 + \hat{z}_n^x \hat{v}_n + \hat{z}_n^x \hat{v}_n$.

Therefore, by (2.19)

$$(2.35) \quad \left| \int_{R \setminus S} (\operatorname{Re} \hat{a}) [|\hat{z}_n|^2 - |\hat{\beta}_n|^2] d\omega \right| < \hat{c}_4 \epsilon^{1/2} \int_R |\hat{z}_n|^2 d\omega$$

where the constant \hat{c}_4 depends on $\hat{\epsilon}$ but not on ϵ . By [2, relation (3.43)]

$$(2.36) \quad \left| \int_{R \setminus S} (\operatorname{Re} \hat{a}) |\hat{z}_n|^2 d\omega \right| < \hat{c}_5 \epsilon^{1/2} \int_R |\hat{z}_n|^2 d\omega ,$$

if n is sufficiently large, and so from (2.35), (2.36)

$$(2.37) \quad \left| \int_{R \setminus S} (\operatorname{Re} \hat{a}) |\hat{\beta}_n|^2 d\omega \right| < \hat{c}_6 \epsilon^{1/2} \int_R |\hat{z}_n|^2 d\omega .$$

The relations (2.28), (2.34), (2.37) and the fact that $\operatorname{Re} \hat{a} > 1/2$, $\omega \in S$, does however yield

$$(2.38) \quad 2^{-1} \int_S |\hat{\beta}_n|^2 d\omega < \hat{c}_6 \epsilon^{1/2} \|\hat{z}_n\|_{L^2(R)}^2 + 2c_2 [\hat{c}_3 \epsilon^{1/2} + \hat{c}] \|\hat{\beta}_n\|_{L^2(R)} \|\hat{z}_n\|_{L^2(R)}.$$

Again recall that $\beta_n = v_n + z_n$ and use (2.19), (2.32), to estimate the right side of (2.38) upwards. The result is

$$(2.39) \quad \int_S |\hat{\beta}_n|^2 d\omega < c_3 [\hat{c}_7 \epsilon^{1/2} + \hat{c}] \|\hat{z}_n\|_{L^2(R)}^2,$$

where c_3 is independent of ϵ, \hat{c} and \hat{c}_7 is independent of ϵ .

In view of (2.39) it is obvious that Lemma 2 holds provided we succeed in showing that

$$(2.40) \quad \int_{R \setminus S} |\hat{\beta}_n|^2 d\omega < \epsilon_0^2 \int_R |\hat{z}_n|^2 d\omega,$$

for any $\epsilon_0 > 0$ if n is sufficiently large. As $\beta_n = v_n + z_n$ then one realizes after recalling (2.19) that (2.40) holds if

$$(2.41) \quad \int_{R \setminus S} |\hat{z}_n|^2 d\omega < \epsilon_0^2 \int_R |\hat{z}_n|^2 d\omega.$$

By assumption $\operatorname{Re} \hat{a}(\omega) > 0$, $\omega \in R \setminus S$. Therefore, for $\tilde{\lambda} > 0$ arbitrary, by [2, relations (3.5), (3.6)]

$$(2.42) \quad \int_{(R \setminus S) \cap [-\tilde{\lambda}, \tilde{\lambda}]} |\hat{z}_n|^2 d\omega < \tilde{f}_1(T, \tilde{\lambda}) \int_R |\hat{z}_n|^2 d\omega$$

where for fixed $\tilde{\lambda}$ the function \tilde{f}_1 can be made arbitrarily small by taking T sufficiently large. Also, by [2, relations (3.20), (3.25), (3.27)]

$$(2.43) \quad \int_{\tilde{\lambda}}^{\infty} + \int_{-\infty}^{-\tilde{\lambda}} |\hat{z}_n|^2 d\omega < c[\tilde{\lambda}^{-2} \int_R |\hat{z}_n|^2 d\omega + \tilde{\lambda}^{-1}],$$

where c is an a priori constant. Fix $\epsilon_0 > 0$. Then take $\tilde{\lambda}$ sufficiently large to make the right side of (2.43) small enough. With $\tilde{\lambda}$ fixed choose T sufficiently large to make the right side of (2.42) small. The relation (2.41) follows and the proof of Lemma 2 is complete.

LEMMA 3. Define, for $i, j = 1, \dots, n$,

$$Y_{ij} = \{y \in L^+(x) | y(\infty) = \alpha_j, y(-\infty) = \alpha_i\}.$$

Then, for $y \in Y_{ij}$,

$$(2.44) \quad y(t) - \alpha_i \in L^2(\mathbb{R}^-)$$

$$(2.45) \quad y(t) - \alpha_j \in L^2(\mathbb{R}^+)$$

$$(2.46) \quad g(y(t)) \in L^2(\mathbb{R}).$$

Proof of Lemma 3. Choose any pair (i, j) such that Y_{ij} is not empty (by Lemma 1 such a Y_{ij} exists) and take any $y \in Y_{ij}$ for which

$$(2.47) \quad h(y(t)) + \alpha_i \notin L^2(\mathbb{R}^-).$$

Define $z(t) = y(t) - \alpha_i$, $\tilde{g}(z) = g(z + \alpha_i)$, $\tilde{h}(z) = \tilde{g}(z) - z$ and rewrite (2.1) as (also use

$$\int_{\mathbb{R}^+} a(s) ds = 1)$$

$$z(t) + \int_{\mathbb{R}^+} \tilde{h}(z(t-s))a(s)ds = 0.$$

Note that as $g(\alpha_i) = 0$ then $\tilde{g}(0) = \tilde{h}(0) = 0$. In addition it is clear that

$$(2.48) \quad \tilde{g}(z) \text{ is locally Lipschitzian, } \liminf_{|z| \rightarrow 0} \left| \frac{\tilde{g}(z)}{z} \right| > 0.$$

Thus we may without loss of generality take $\alpha_i = 0$.

Let $t_n \rightarrow \infty$. We claim that there exists $\rho > 0$ such that

$$(2.49) \quad \sup_{-\infty < s < 0} \int_{s-2t_n}^s |h(z(\tau))|^2 d\tau < \rho \int_{-2t_n}^0 |h(z(\tau))|^2 d\tau.$$

Suppose not. Then there exists $T_n \rightarrow \infty$ for which

$$(2.50) \quad \sup_{-\infty < s < 0} \int_{s-2t_n}^s |h(z(\tau))|^2 d\tau < 2 \int_{-T_n-2t_n}^{-T_n} |h(z(\tau))|^2 d\tau,$$

and so

$$(2.51) \quad \sup_{-\infty < s < -T_n} \int_{s-2t_n}^s |h(z(\tau))|^2 d\tau < 2 \int_{-T_n-2t_n}^{-T_n} |h(z(\tau))|^2 d\tau.$$

Define $\tilde{z}_n(t) = z(t - t_n - T_n)$, $t \in \mathbb{R}$. Then (2.51) is equivalent to

$$(2.52) \quad \sup_{-\infty < s < t_n} \int_{s-2t_n}^s |h(\tilde{z}_n(\tau))|^2 d\tau < 2 \int_{-t_n}^{t_n} |h(\tilde{z}_n(\tau))|^2 d\tau.$$

From Lemma 2, (2.47) and (2.52) follows

$$(2.53) \quad \int_{-t_n}^{t_n} |g(\tilde{z}_n(\tau))|^2 d\tau < \epsilon_0^2 \int_{-t_n}^{t_n} |h(\tilde{z}_n(\tau))|^2 d\tau,$$

for $\epsilon_0 > 0$ arbitrary if n is sufficiently large. But

$$|h(y)|^2 < 2|g(y)|^2 + 2|y|^2 < 2(\lambda^2 + 1)|y|^2 \quad \text{and so by (2.53)}$$

$$(2.54) \quad \int_{-t_n}^{t_n} |g(\tilde{z}_n(\tau))|^2 d\tau < \epsilon_0^2 [2\lambda^2 + 2] \int_{-t_n}^{t_n} |\tilde{z}_n(\tau)|^2 d\tau.$$

Also, as $\tilde{z}_n(t_n) = z(-T_n)$ and as $z(-\infty) = 0$ then $\lim_{n \rightarrow \infty} \tilde{z}_n(\tau) = 0$, uniformly on $[-t_n, t_n]$. This fact, together with (1.14) clearly gives

$$(2.55) \quad \rho \int_{-t_n}^{t_n} |\tilde{z}_n(\tau)|^2 d\tau < \int_{-t_n}^{t_n} |g(\tilde{z}_n(\tau))|^2 d\tau$$

for some constant $\rho > 0$. From (2.54), (2.55) follows $\sup_n \int_{-t_n}^{t_n} |\tilde{z}_n(\tau)|^2 d\tau < \infty$.

Consequently the right side of (2.52) is uniformly bounded in n , hence the same is true

for the right side of (2.50) and so the left side of (2.50) is uniformly bounded in n .

But this violates (2.47) and thus (2.49) must hold.

Let $z_n(\tau) \stackrel{\text{def}}{=} z(\tau - t_n)$. Then, by (2.49)

$$\sup_{-\infty < s < t_n} \int_{s-2t_n}^s |h(z_n(\tau))|^2 d\tau < \rho \int_{-t_n}^{t_n} |h(z_n(\tau))|^2 d\tau$$

and so, by (2.47), we may apply Lemma 2. Thus $\int_{-t_n}^{t_n} |g(z_n(\tau))|^2 d\tau < \epsilon_0^2 \int_{-t_n}^{t_n} |h(z_n(\tau))|^2 d\tau$ or equivalently,

$$\int_{-2t_n}^0 |g(z(\tau))|^2 d\tau < \epsilon_0^2 \int_{-2t_n}^0 |h(z(\tau))|^2 d\tau.$$

But again, as $z(-\infty) = 0$ and by the second part of (2.48) and as $|h(z)| < K|z|$, this gives $\sup_n \int_{-2t_n}^0 |z(\tau)|^2 d\tau < \infty$. We conclude that (2.44) is satisfied.

To get (2.45) (and (2.46)) one proceeds as follows. For simplicity let

$\alpha_1 = 0$, $\alpha_j \neq 0$. Define $z(t) = y(t) - \alpha_j$, $\tilde{g}(z) = g(z + \alpha_j)$, $\tilde{h}(z) = \tilde{g}(z) - z$. The equation (2.1) considered on R^+ can be written

$$(2.56) \quad z(t) + \int_0^t \tilde{h}(z(t-s)) a(s) ds = F(t), \quad t \in R^+,$$

where $F(t) = - \int_t^\infty h(y(t-s)) a(s) ds - \alpha_j \int_t^\infty a(s) ds$. A differentiation of (2.56) gives

$$z'(t) + \int_{[0,t]} \tilde{h}(z(t-s)) da(s) = F'(t), \quad \text{a.e. on } R^+,$$

with $F'(t) = - \int_{[t,\infty)} h(y(t-s)) da(s) + \alpha_j a(t)$. By (2.44), as α is finite and as $a \in L^2(R^+)$ one has

$$(2.57) \quad F' \in L^2(R^+).$$

By essentially repeating the estimates of [2, Theorem 2] and those of Lemma 2 above but replacing (2.8) and (2.26) by the much stronger assumption (2.57) one obtains, for any $\epsilon_0 > 0$, if t is sufficiently large,

$$(2.58) \quad \int_0^t |\tilde{g}(z(\tau))|^2 d\tau < \epsilon_0^2 \int_0^t |\tilde{h}(z(\tau))|^2 d\tau.$$

But (2.58) combined with (1.8) and (1.14) immediately gives (2.45).

LEMMA 4. Let $y \in Y_{1,j}$ for some i, j ; $g(y(t)) \not\equiv 0$, and define $G(x) = \int_0^x g(u) du$, $x \in R$. Then

$$(2.59) \quad G(y_j) < G(y_1).$$

Proof of Lemma 4. Let $r_\lambda \in LAC(R^+)$, $\lambda > 0$, be the solution of

$$(2.60) \quad r'_\lambda(t) + \lambda(r_\lambda * \mu)(t) = 0 \quad \text{a.e. on } R^+, \quad r_\lambda(0) = 1.$$

By (1.5), (1.7) and by [2, Lemmas 2 and 3]

$$(2.61) \quad r'_\lambda \in (L^1 \cap NBV)(\mathbb{R}^+), \quad \lambda > 0,$$

$$(2.62) \quad \lim_{t \rightarrow \infty} r'_\lambda(t) = \lim_{t \rightarrow \infty} (r'_\lambda * f)(t) = 0, \quad \lambda > 0.$$

Convolve (1.1) by r'_λ and use (2.60)-(2.62). This yields that any $y \in L^+(x)$ satisfies

$$(2.63) \quad y(t) + \int_0^\infty \left[\frac{g(y(t-s))}{\lambda} - y(t-s) \right] a_\lambda(s) ds = 0, \quad t \in \mathbb{R},$$

where $a_\lambda(t) \stackrel{\text{def}}{=} -r'_\lambda(t)$. Differentiate (2.63) to obtain

$$(2.64) \quad y'(t) + \int_0^\infty \left[\frac{g(y(t-s))}{\lambda} - y(t-s) \right] da_\lambda(s) = 0, \quad \text{a.e. on } \mathbb{R}$$

where $a_\lambda([0, t]) = a_\lambda(t)$.

Fix i, j and take any $y \in Y_{ij}$, $g(y(t)) \not\equiv 0$. Let $\varepsilon > 0$ be arbitrary and observe that as $y(\infty) = y_j$, $g(y_j) = 0$, $a_\lambda \in L^1(\mathbb{R}^+)$ then given $\lambda > 0$ there exists $0 < T_\lambda < \infty$ such that if $y_\lambda(s) \stackrel{\text{def}}{=} y(s + T_\lambda)$, $s \in \mathbb{R}$, then

$$(2.65) \quad \int_{\mathbb{R}^+} \left[\frac{g(y_\lambda(s))}{\lambda} + (y_j - y_\lambda(s)) \right] a_\lambda(s) ds < \frac{\varepsilon}{2}.$$

For each $\lambda > 0$ choose $T_\lambda > 0$ such that (2.65) holds. Then define

$$(2.66) \quad f_\lambda(s) = \begin{cases} \lambda^{-1} g(y_\lambda(s)) - y_\lambda(s) + y_i, & s < 0, \\ \lambda^{-1} g(y_\lambda(s)) - y_\lambda(s) + y_j, & s > 0, \end{cases}$$

$$h_\lambda(s) = \begin{cases} 0, & s < 0, \\ [y_j - y_i] a_\lambda(s), & s > 0. \end{cases}$$

These definitions, (2.64) and $a_\lambda(\mathbb{R}^+) = 0$ yield

$$(2.67) \quad y'_\lambda(t) + \int_0^\infty f_\lambda(t-s) d\alpha_\lambda(s) = h_\lambda(t), \quad \text{a.e. } t \in \mathbb{R}, \quad \lambda > 0.$$

Note that by Lemma 3 $f_\lambda \in L^2(\mathbb{R})$ and so, by (2.61), (2.67) and as $h_\lambda \in L^2(\mathbb{R})$ one has $y'_\lambda \in L^2(\mathbb{R})$. Thus the integrals below are well defined.

Multiply (2.67) by f_λ , integrate over \mathbb{R} and apply Parseval's relation. This gives

$$(2.68) \quad \int_{\mathbb{R}} y'_\lambda(\tau) f_\lambda(\tau) d\tau + (2\pi)^{-1} \int_{\mathbb{R}} |\hat{f}_\lambda|^2 \operatorname{Re} \hat{\alpha}_\lambda d\omega = \int_{\mathbb{R}^+} f_\lambda(\tau) h_\lambda(\tau) d\tau.$$

Simple calculations result in

$$(2.69) \quad \int_{\mathbb{R}} y'_\lambda(\tau) f_\lambda(\tau) d\tau = \lambda^{-1} [G(y_j) - G(y_i)] + 2^{-1} [y_j^2 - y_i^2] - y_\lambda(0) [y_j - y_i].$$

By (2.65), (2.68), (2.69)

$$(2.70) \quad |G(y_j) - G(y_i)| + \lambda(2\pi)^{-1} \int_{\mathbb{R}} |\hat{f}_\lambda|^2 \operatorname{Re} \hat{\alpha}_\lambda d\omega < \varepsilon$$

provided λ is taken small enough so that

$$\lambda[2^{-1} [y_j^2 - y_i^2] - y_\lambda(0) [y_j - y_i]] < 2^{-1} \varepsilon.$$

The relation (2.70) does however imply that (2.59) is a consequence of

$$(2.71) \quad \liminf_{\lambda \rightarrow 0} \lambda(2\pi)^{-1} \int_{\mathbb{R}} |\hat{f}_\lambda|^2 \operatorname{Re} \hat{\alpha}_\lambda d\omega > 0,$$

which we intend to prove.

Note at first that

$$\operatorname{Re} \hat{\alpha}_\lambda(\omega) = \lambda \omega^2 (\operatorname{Re} \hat{\mu}(\omega)) |i\omega + \lambda \hat{\mu}(\omega)|^{-2}, \quad \omega \neq 0, \\ \hat{\alpha}_\lambda(0) = 0$$

and so $\operatorname{Re} \hat{\alpha}_\lambda(\omega) > 0$, $\omega \in \mathbb{R}$. By (1.13) $\operatorname{Re} \hat{\alpha}_\lambda(\omega) = 0$ if and only if

$\omega \in \{0\} \cup \{|\omega| |\hat{\mu}(\omega)| = \infty\}$. In addition one has $m(\{|\omega| |\hat{\mu}(\omega)| = \infty\}) = 0$. See [3, p. 230],

and so $\operatorname{Re} \hat{\alpha}_\lambda(\omega) > 0$ a.e. on \mathbb{R} . Let z_λ be defined by

$$(2.72) \quad z_\lambda(s) = \begin{cases} y_i - y_\lambda(s), & s < 0, \\ y_j - y_\lambda(s), & s > 0, \end{cases} \quad \lambda > 0,$$

thus $f_\lambda(s) = \lambda^{-1}g(y_\lambda(s)) + z_\lambda(s)$, $s \in \mathbb{R}$. Also let

$$u(s) \stackrel{\text{def}}{=} \begin{cases} y_1 - y(s), & s < 0, \\ y_j - y(s), & s > 0, \end{cases} \quad u_\lambda(s) \stackrel{\text{def}}{=} u(s + T_\lambda), \quad s \in \mathbb{R}.$$

Then

$$z_\lambda(s) - u_\lambda(s) = \begin{cases} 0 & s < -T_\lambda \\ y_1 - y_j & -T_\lambda \leq s \leq 0 \\ 0 & s > 0, \end{cases}$$

and

$$|\hat{z}_\lambda(\omega)|^2 \leq 2|\hat{u}_\lambda(\omega)|^2 + 2|\hat{z}_\lambda - u_\lambda(\omega)|^2 \leq 2|\hat{u}(\omega)|^2 + 4|\omega|^{-2}|y_1 - y_j|^2$$

which implies

$$(2.73) \quad \lambda \int_{\mathbb{R}} |\hat{z}_\lambda(\omega)|^2 \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega \leq 2\lambda \int_{\mathbb{R}} |\hat{u}(\omega)|^2 \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega + 4\lambda|y_1 - y_j|^2 \int_{\mathbb{R}} \omega^{-2} \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega.$$

Observe that u is independent of λ .

By (2.6) and as $a \in L^1(\mathbb{R}^+)$ we have

$$\lim_{|\omega| \rightarrow \infty} \frac{|\hat{\mu}(\omega)|}{|i\omega + \mu(\omega)|} = 0,$$

and therefore there exist constants ω_1, K_1 such that provided $|\omega| > \omega_1$ then

$$(2.74) \quad |\hat{\mu}(\omega)| < K_1 |\omega|.$$

But by (2.3), (2.74) and as $|\hat{\alpha}| < K_2$, for some K_2 , $\omega \in \mathbb{R}$,

$$(2.75) \quad 0 < \operatorname{Re} \hat{\mu}(\omega) \leq K_2 \omega^{-2} |i\omega + \hat{\mu}(\omega)|^2 \leq K_2 (K_1 + 1)^2 < \infty$$

for $|\omega| > \omega_1$. By (2.4), for $\omega > 0$,

$$\omega^2 |\operatorname{Im} \hat{\mu}(\omega)| |i\omega + \hat{\mu}(\omega)|^{-2} \leq K_2 + \omega |\hat{\mu}(\omega)|^2 |i\omega + \hat{\mu}(\omega)|^{-2},$$

and so, using (2.74), (2.75)

$$(2.76) \quad |\operatorname{Im} \hat{\mu}(\omega)| \leq K_3 + \omega^{-1} |\operatorname{Im} \hat{\mu}(\omega)|^2, \quad |\omega| > \omega_1,$$

for some constant K_3 . Let $\omega_n \rightarrow \infty$ be such that $|\operatorname{Im} \hat{\mu}(\omega_n)| \rightarrow \infty$. Then, by (2.76),

$$\omega_n |\operatorname{Im} \hat{\mu}(\omega_n)| \leq 2 |\operatorname{Im} \hat{\mu}(\omega_n)|^2$$

and consequently

$$(2.77) \quad |\operatorname{Im} \hat{\mu}(\omega_n)| > 2^{-1} \omega_n.$$

But if (2.75), (2.77) are used in (2.6) then $\liminf_{n \rightarrow \infty} |\hat{a}(\omega_n)| > 0$ follows. This however violates $a \in L^1(\mathbb{R}^+)$. Hence $\limsup_{|\omega| \rightarrow \infty} |\operatorname{Im} \hat{\mu}(\omega)| < \infty$ and so, for some γ_1, γ_2 ,

$$|\hat{\mu}(\omega)| < \gamma_2, \quad |\omega| < \gamma_1.$$

Without loss of generality take γ_1 such that

$$(2.78) \quad |i\omega| |i\omega + \hat{\mu}(\omega)|^{-1} > 2^{-1}, \quad |\omega| > \gamma_1.$$

As $\hat{\mu}(\omega)[i\omega + \hat{\mu}(\omega)]^{-1} \in L^\infty(\mathbb{R})$ and by (1.13) we have

$$(2.79) \quad \lambda \operatorname{Re} \hat{\alpha}_\lambda(\omega) < \omega^2 [\operatorname{Re} \hat{\mu}(\omega)]^{-1} < \infty, \quad |\omega| < \gamma_1,$$

$$(2.80) \quad \lambda \omega^{-2} \operatorname{Re} \hat{\alpha}_\lambda(\omega) < [\operatorname{Re} \hat{\mu}(\omega)]^{-1} < \infty, \quad |\omega| < \gamma_1.$$

In addition note that $\hat{\alpha}_\lambda = (1 - \lambda) \hat{\alpha} + \lambda \hat{\alpha}_\lambda$ and so

$$(2.81) \quad |\hat{\alpha}_\lambda| < c_1 |\hat{\alpha}| + \lambda c_2, \quad \omega \in \mathbb{R}, \quad \lambda > 0,$$

for some constants c_1, c_2 . But

$$(2.82) \quad \hat{\alpha}_\lambda(\omega) = \left[\frac{\lambda \hat{\mu}(\omega)}{i\omega + \hat{\mu}(\omega)} \right] \left[\frac{i\omega}{i\omega + \hat{\mu}(\omega)} + \frac{\lambda \hat{\mu}(\omega)}{i\omega + \hat{\mu}(\omega)} \right]^{-1}, \quad \omega \in \mathbb{R},$$

and clearly

$$(2.83) \quad \lim_{\lambda \rightarrow 0} [\lambda \hat{\mu}(\omega)] [i\omega + \hat{\mu}(\omega)]^{-1} = 0, \quad \text{uniformly for } \omega \in \mathbb{R}.$$

From (2.78), (2.81)-(2.83) follows

$$(2.84) \quad \lim_{\lambda \rightarrow 0} \hat{\alpha}_\lambda(\omega) = 0, \quad \text{uniformly on } |\omega| > \gamma_1.$$

Also, if $\omega \neq 0$, $\omega \notin \{\omega | |\hat{\mu}(\omega)| = \infty\}$ then $|i\omega| |i\omega + \hat{\mu}(\omega)|^{-1} \neq 0$ and hence by (2.81),

$$(2.82)$$

$$(2.85) \quad \lim_{\lambda \rightarrow 0} \operatorname{Re} \hat{\alpha}_\lambda(\omega) = 0 \quad \text{a.e. on } [-\gamma_1, \gamma_1].$$

Now write

$$\lambda \int_{\mathbb{R}} |u|^2 \operatorname{Re} \hat{\alpha}_\lambda d\omega = \lambda \int_{-\gamma_1}^{\gamma_1} + \lambda \int_{\gamma_1}^{\infty} + \lambda \int_{-\infty}^{-\gamma_1} \{|u|^2 \operatorname{Re} \hat{\alpha}_\lambda\} d\omega,$$

and let $\lambda \rightarrow 0$. By (2.79), (2.85), as $u \in L^2(\mathbb{R})$ and by the dominated convergence theorem

the first integral on the right tends to zero. From (2.84) and again using $u \in L^2(\mathbb{R})$ it follows that the two remaining integrals vanish as $\lambda \rightarrow 0$. Thus

$$(2.86) \quad \lim_{\lambda \rightarrow 0} \lambda \int_{\mathbb{R}} |u|^2 \operatorname{Re} \hat{\alpha}_\lambda d\omega = 0.$$

By (2.80), (2.84), (2.85) and by the dominated convergence theorem

$$(2.87) \quad \lim_{\lambda \rightarrow 0} \lambda \int_{\mathbb{R}} \omega^{-2} \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega = 0.$$

Thus, from (2.73), (2.86), (2.87)

$$(2.88) \quad \lim_{\lambda \rightarrow 0} \lambda \int_{\mathbb{R}} |\hat{z}_\lambda(\omega)|^2 \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega = 0.$$

Write $\hat{g}_\lambda(\omega) = \widehat{g(y_\lambda(s))}(\omega)$, $\hat{g}(\omega) = \widehat{g(y(s))}(\omega)$ and consider the expression

$$\lambda \int_{\mathbb{R}} \left| \frac{\hat{g}_\lambda(\omega)}{\lambda} \right|^2 \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega$$

which, as $|\hat{g}_\lambda(\omega)| = |\hat{g}(\omega)|$, equals $\lambda^{-1} \int_{\mathbb{R}} |\hat{g}(\omega)|^2 \operatorname{Re} \hat{\alpha}_\lambda(\omega) d\omega$. If

$\omega \notin \{0\} \cup \{\omega \mid |\hat{\mu}(\omega)| = \infty\}$ then

$$(2.89) \quad \lim_{\lambda \rightarrow 0} \lambda^{-1} \operatorname{Re} \hat{\alpha}_\lambda(\omega) = \lim_{\lambda \rightarrow 0} [\omega^2 \operatorname{Re} \hat{\mu}(\omega)] |i\omega + \lambda \hat{\mu}(\omega)|^{-2} = \operatorname{Re} \hat{\mu}(\omega).$$

Take any finite interval $[-T, T] \subset \mathbb{R}$ such that for some $\varepsilon > 0$

$$\int_{-T}^T |\hat{g}(\omega)|^2 d\omega > 3\varepsilon$$

and then any $\delta > 0$ such that if $S \subset [-T, T]$, $m(S) < \delta$, then

$$\int_S |\hat{g}(\omega)|^2 d\omega < \varepsilon.$$

By Egoroff's theorem and by (2.89) there exists $E \subset [-T, T]$, $m(E) < \delta$, for which the convergence in (2.89) is uniform on $[-T, T] \setminus E$. Then, as $\operatorname{Re} \hat{\alpha}_\lambda(\omega) > 0$, $\omega \in \mathbb{R}$,

$$(2.90) \quad \left\{ \begin{array}{l} \int_R \lambda^{-1} |\hat{g}(\omega)|^2 \operatorname{Re} \hat{a}_\lambda(\omega) d\omega > \int_{[-T, T] \setminus E} \lambda^{-1} |\hat{g}(\omega)|^2 \operatorname{Re} \hat{a}_\lambda(\omega) d\omega \\ > \rho 2^{-1} \int_{[-T, T] \setminus E} |\hat{g}(\omega)|^2 d\omega > \epsilon \rho > 0, \end{array} \right.$$

provided λ is taken sufficiently small and where $\rho = \inf_{|\omega| < T} \operatorname{Re} \hat{\mu}(\omega)$.

From (2.66), (2.72), (2.88), (2.90) we get (2.71) after straightforward estimates.

Thus (2.59) holds and Lemma 4 is proved.

With (2.59) at our disposal we are ready to prove the theorem.

Suppose (1.15) does not hold and define

$$X_0 \stackrel{\text{def}}{=} \{\alpha_1 \in X \mid \alpha_1 \in L^+(x)\}.$$

By Lemma 1 X_0 is not empty. Let $\alpha \in X_0$ be such that

$$(2.91) \quad G(\alpha) = \min_{\alpha_1 \in X_0} G(\alpha_1).$$

Take any sequence $t_n \rightarrow \infty$ satisfying $x(t + t_n) \rightarrow \alpha$ uniformly on compact sets and define $x_n(t) = x(t + t_n)$, $t \in \mathbb{R}$. Let $d = \operatorname{dist}(\alpha, X_0 \setminus \alpha) > 0$. Take $\delta \in (0, 2^{-1}d]$ such that $T_n \stackrel{\text{def}}{=} \inf\{\tau \mid \tau > 0, |x_n(\tau) - \alpha| = \delta\}$ is well-defined for all n . Let $\tilde{x}_n(t) \stackrel{\text{def}}{=} x_n(t + T_n)$.

As a consequence of Lemma 1 there exists a subsequence $\{\tilde{x}_{n_k}\}$ of $\{\tilde{x}_n\}$ and $y(t)$ such that

$$\tilde{x}_{n_k}(t) \rightarrow y(t), \text{ uniformly on compact sets,}$$

and for which $y(-\infty) = \alpha$, $y(\infty)$ exists and $\in X_0$. But by (2.59) $G(y(\infty)) - G(y(\alpha)) < 0$.

This however violates (2.91) and completes the proof of the theorem.

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20. ABSTRACT - cont'd.

in the case when f only satisfies $\lim_{t \rightarrow \infty} (r * f)(t) = 0$. The function $r(t)$ is defined as the solution of

$$r'(t) + \int_{[0,t]} r(t-s) d\mu(s) = 0, \quad t \in \mathbb{R}^+, \quad r(0) = 1,$$

and we assume $r' \in (L^1 \cap NBV)(\mathbb{R}^+)$. No sign condition is imposed on $g(x)$.

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